Running Spectral Index in Noncommutative Inflation and WMAP Three Year Results

Qing-Guo Huang and Miao Li

Institute of Theoretical Physics, Academia Sinica
P. O. Box 2735, Beijing 100080

The interdisciplinary center of theoretical Studies, Academia Sinica
P. O. Box 2735, Beijing 100080

huangqg@itp.ac.cn
mli@itp.ac.cn

A model independent analysis shows that the running of the spectral index of the three year WMAP results can be nicely realized in noncommutative inflation. We also re-examine some concrete noncommutative inflation models. We find that a large tensor-scalar ratio is required, corresponding to a low number of e-folds before the end of inflation in some simple models.

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An epoch of accelerated expansion in the early universe, inflation, dynamically resolves many cosmological puzzles such as homogeneity, isotropy and flatness of the universe [1], and generates superhorizon fluctuations without appealing to fine-tuned initial setups. These fluctuations become classical after crossing out the Hubble horizon. During the deceleration phase after inflation they re-enter the horizon, and seed the matter and the radiation fluctuations observed in the universe. The anisotropy in CMB encodes the very important information for inflation.

The ΛCDM model remains an excellent fit to the three years WMAP data and other astronomical data [2]. In the simplest models for the structure formation, a scale-invariant spectrum of the primordial power spectrum is no longer a good fit to the three year data, implying that inflation must be dynamic. Some simple inflation models are already ruled out [2,3]. The deficit of power in low mulitipoles of the angular power spectrum in the first year results of WMAP is not preferred by the three years results anymore. Even though allowing for a running spectral index slightly improves the fit to the WMAP data, the improvement in the fit is not significant enough to require this new parameter. However, it is important that a running index still survives the three year data together with other sets of data. Since it is hard to realize a large running in the usual slow-roll inflation models [4], we hope that future experiments will confirm a running spectral index and thus open a new window into the physics of the first moments of the big bang and provides some clues into trans-Planckian physics.

Noncommutative spacetime naturally emerges in string theory [5], which implies a new uncertainty relation

$$\Delta t_p \Delta x_p \geq l_s^2, \tag{1}$$

where t_p and x_p are the physical time and space, l_s is uncertainty length scale or string scale in string theory. Constructing a realistic inflation model in noncommutative spacetime is still an open question. Motivated by the first year WMAP resuls, we found in [7-9] that a toy noncommutative inflation model [6] can accommodate a large running, this model was later extensively studied in [10]. Particularly in [9] we found that the noncommutative effects always make the power spectrum more blue and the noncommutative effects on the small scale fluctuations can be ignored, which is consistent with observational results. Other models with a large running are discussed in [11-13]. We re-examine the model studied in [7-9] in the light of the three year WMAP results.

The spacetime noncommutative effects are encoded in a new product among functions, namely the star product, replacing the usual algebra product. The evolution of the

background is homogeneous and the standard cosmological equations of the inflation will not change and still take the form in Friedmann-Robertson-Walker (FRW) Universe:

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0, \tag{2}$$

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{3M_p^2}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right), \tag{3}$$

here M_p is the reduced Planck mass and we assumed the universe be spatially flat and the inflaton ϕ be spatially homogeneous. If $\dot{\phi}^2 \ll V(\phi)$ and $\ddot{\phi} \ll 3H\dot{\phi}$, the scalar field slowly rolls down its potential. Define some slow-roll parameters,

$$\epsilon_V = \frac{M_p^2}{2}\left(\frac{V'}{V}\right)^2, \tag{4}$$

$$\eta_V = M_p^2 \frac{V''}{V}, \tag{5}$$

$$\xi_V = M_p^4 \frac{V'V'''}{V^2}. \tag{6}$$

The slow-roll condition can be expressed as $\epsilon_V, \eta_V \ll 1$.

In order to make the uncertainty relationship in (1) more clear in the Friedmann-Robertson-Walker (FRW) background, we introduce another time coordinate τ in the noncommutative spacetime such that the metric takes the form

$$ds^2 = dt^2 - a^2(t)d\vec{x}^2 = a^{-2}(\tau)d\tau^2 - a^2(\tau)d\vec{\tau}^2. \tag{7}$$

Now the uncertainty relationship (1) becomes

$$\Delta\tau\Delta x \geq l_s^2. \tag{8}$$

The star product can be explicitly defined as

$$f(\tau, x) * g(\tau, x) = e^{-\frac{i}{2}l_s^2(\partial_x\partial_{\tau'} - \partial_\tau\partial_{x'})} f(\tau, x)g(\tau', x')|_{\tau'=\tau, x'=x}. \tag{9}$$

Since the comoving curvature perturbation $\mathcal{R}$ depends on the space and time, the equation of motion for $\mathcal{R}$ is modified by the noncommutative effects

$$u_k'' + \left(k^2 - \frac{z_k''}{z_k}\right)u_k = 0, \tag{10}$$

where

$$
\begin{aligned}
&z_k^2(\tilde{\eta}) = z^2 y_k^2(\tilde{\eta}), \quad y_k^2 = (\beta_k^+ \beta_k^-)^{\frac{1}{2}}, \\
&\frac{d\tilde{\eta}}{d\tau} = \left(\frac{\beta_k^-}{\beta_k^+}\right)^{\frac{1}{2}}, \quad \beta_k^\pm = \frac{1}{2}(a^{\pm 2}(\tau + l_s^2 k) + a^{\pm 2}(\tau - l_s^2 k)),
\end{aligned} \tag{11}
$$

here l_s is the string length scale, $z = a\dot{\phi}/H$, $\mathcal{R}_k(\tilde{\eta}) = u_k(\tilde{\eta})/z_k(\tilde{\eta})$ is the Fourier modes of $\mathcal{R}$ in momentum space and the prime denotes derivative with respect to the modified conformal time $\tilde{\eta}$. The deviation from the commutative case encodes in $\beta_k^\pm$ and the corrections from the noncommutative effects can be parameterized by $\frac{Hk}{aM_s^2}$. After a lengthy but straightforward calculation, we get

$$
\begin{aligned}
&\frac{z_k''}{z_k} = 2(aH)^2\left(1 + \frac{5}{2}\epsilon_V - \frac{3}{2}\eta_V - 2\mu\right), \\
&aH \simeq \frac{-1}{\tilde{\eta}}(1 + \epsilon_V + \mu),
\end{aligned} \tag{12}
$$

where $\mu = H^2k^2/(a^2M_s^4)$ is the noncommutative parameter and $M_s = l_s^{-1}$ is the string mass scale. Solving eq. (10) yields the amplitude of the scalar comoving curvature fluctuations in noncommutative spacetime

$$
\Delta_{\mathcal{R}}^2 \simeq \frac{k^3}{2\pi^2}\left|\mathcal{R}_k(\tilde{\eta})\right|^2 = \frac{V/M_p^4}{24\pi^2\epsilon_V}(1+\mu)^{-4-6\epsilon_V+2\eta_V}, \tag{13}
$$

where H and V take the values when the fluctuation mode k crosses the Hubble radius $(z_k''/z_k = k^2)$, k is the comoving Fourier mode. Plugging this condition into eq. (12), we obtain

$$
\begin{aligned}
&d\ln k = (1 - \epsilon_V + 4\epsilon_V\mu)H dt, \\
&\frac{d\mu}{d\ln k} = (1 + \epsilon_V - 4\epsilon_V\mu)\frac{1}{H}\frac{d}{dt}\left(\frac{H^2k^2}{a^2M_s^4}\right) \simeq -4\epsilon_V\mu.
\end{aligned} \tag{14}
$$

Using eq. (13) and (14), we obtain the index of the power spectrum for the scalar fluctuations and its running respectively

$$
n_s - 1 \equiv s = \frac{d\ln\Delta_{\mathcal{R}}^2}{d\ln k} = -6\epsilon_V + 2\eta_V + 16\epsilon_V\mu, \tag{15}
$$

$$
\frac{dn_s}{d\ln k} \equiv \alpha_s = -24\epsilon_V^2 + 16\epsilon_V\eta_V - 2\xi_V - 32\epsilon_V\eta_V\mu. \tag{16}
$$

The tensor-scalar ratio is

$$
r = 16\epsilon_V. \tag{17}
$$

Here we only sketch out the brief deviation of the primordial power spectrum and the spectral index and its running. See [9] in details. With WMAP data only, the best fit results for the ΛCDM model with running and tensors are

$$\begin{aligned} n_s = 1.21^{+0.13}_{-0.16}, \quad dn_s/d\ln k = -0.102^{+0.050}_{-0.043}, \\ \Delta^2_{\mathcal{R}}(k = 0.05Mpc^{-1}) = \left(20.9^{+1.3}_{-1.9}\right) \times 10^{-10}, \end{aligned} \tag{18}$$

and

$$r \leq 1.5 \quad \text{at } 95\% \text{ CL}. \tag{19}$$

We need to stress that the improvement in the fit is not significant enough to require a new parameter, even through a running spectral index slightly improves the fit to the WMAP data. Since ϵ_V must be positive, eq. (15) says that the noncommutative effects always make the spectral index more blue. If η_V is also positive, the noncommutative effects can help us to fit the spectral index and its running nicely.

From eq.(15), we have $\eta_V = \frac{1}{2}\left(s + \frac{3}{8}r - r\mu\right)$. With this substituted into eq.(16), it becomes

$$\frac{dn_s}{d\ln k} = \frac{3}{32}r^2 + \frac{s}{2}r - r\left(\frac{7}{8}r + s\right)\mu + r^2\mu^2 - 2\xi_V. \tag{20}$$

If spacetime is commutative ($\mu = 0$), eq.(20) becomes

$$\frac{dn_s}{d\ln k} = \frac{3}{32}r^2 + \frac{s}{2}r - 2\xi_V. \tag{21}$$

Thus in the commutative case, ξ_V must be large in order to get a large enough negative value of $dn_s/d\ln k$ in (21), when the CMB power spectrum is blue ($s > 0$), and $r > 0$. However it is quite hard to have a large ξ_V in the known typical inflation model [4]. If we take the noncommutative effects into account, the second or the third term on the right hand side of eq.(20) will become negative for some suitable values of μ, there is no longer the need for a large ξ_V.

Ignoring ξ_V, the running α_s becomes negative provided

$$r > \frac{\mu s - \frac{s}{2}}{\frac{3}{32} - \frac{7}{8}\mu + \mu^2}, \tag{22}$$

where we assumed that the denominator is negative, this is the case when r is in the range $\frac{1}{8} < \mu < \frac{3}{4}$. If $\mu > \frac{1}{2}$, and s is positive (which is the case at $k = 0.002(\text{Mpc})^{-1}$), the lower bound for r is negative thus in this case the running is always negative. For a more reasonable value of μ, $\mu < \frac{1}{2}$, the lower bound for r is positive. As an example, taking

$s = 0.2$, $\mu = \frac{1}{4}$, the lower bound for r is 0.8. We see that indeed for a small μ, a large r is required to have a negative running.

Since the value of ξ_V depends on the concrete inflation model, we shall ignore ξ_V which is positive in many typical inflation models in order to make a model-independent analysis and show the constraint on values of $r = 16\epsilon_V$ and of μ in order to fit the experimental data, in fig. 1.

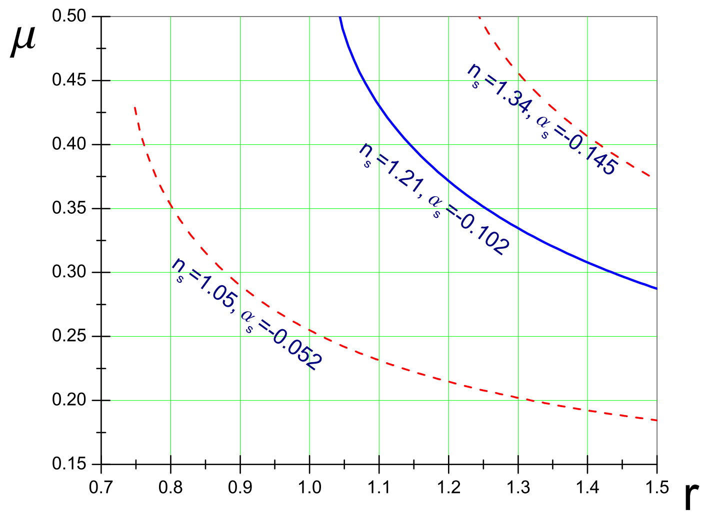

Figure 1. *The value for the blue solid line is* $n_s = 1.21, dn_s/d\ln k = -0.102$. *The tensor-scalar ratio* r *is not greater than 1.5 at* 95% *CL. The range between these two red dash lines is permitted. Here we neglect* ξ_V.

Large enough tensor-scalar ratio is needed ($r \geq 0.75$) in order to fit the running of the spectral index for WMAP. The permitted range for the parameter is nice and our results can be trusted.

- *Chaotic inflation*

Chaotic inflation is drived by the scalar field with potential $V(\phi) \sim \phi^n$. The slow roll parameters are related to the number of e-folds N before the end of the inflation by

$$\begin{aligned} \epsilon_V &= \frac{n}{4N}, \\ \eta_V &= \frac{n-1}{2N}, \\ \xi_V &= \frac{(n-1)(n-2)}{4N^2}. \end{aligned} \tag{23}$$

The spectral index and its running and the tensor-scalar ratio are given by

$$\begin{aligned}
n_s &= 1 + \left(\mu - \frac{1}{8} - \frac{1}{4n}\right) r, \\
\alpha_s &= -\frac{n+2}{32n^2} r^2 \left(1 + \frac{8n(n-1)}{n+2}\mu\right), \\
r &= \frac{4n}{N}.
\end{aligned} \tag{24}$$

The running of the spectral index is always negative for the chaotic inflation model. The power spectrum is always a red spectrum in the commutative spacetime. But it becomes blue if $\mu \geq \frac{1}{8} + \frac{1}{4n}$ in noncommutative chaotic inflation. In the following we check two special cases with $n = 2$ and $n = 4$.

1) For n=2.

The spectral index and its running and the tensor-scalar ratio are

$$\begin{aligned}
n_s &= 1 + \left(\mu - \frac{1}{4}\right) r, \\
\alpha_s &= -\frac{r^2}{32} (1 + 4\mu), \\
r &= \frac{8}{N}.
\end{aligned} \tag{25}$$

If we take $14 \leq N \leq 75$ [3,14], $0.11 \leq r \leq 0.57$ and $-1.0 \times (1 + 4\mu) \times 10^{-2} \leq \alpha_s \leq -3.8 \times (1 + 4\mu) \times 10^{-4}$. In the commutative spacetime $\mu = 0$, the running of the spectral index satisfies $\alpha_s \in (-10^{-2}, -10^{-4})$. If we take $N = 14$ to fit $n_s = 1.21$, we find $r = 0.57$, $\mu = 0.618$ and $\alpha_s = -3.5 \times 10^{-2}$. Usually we cannot take so small number of e-folds. A more reasonable number of e-folds related to the observable perturbations is $N = 50$. In order to make the expansion for the noncommutative parameter trusted, we take $\mu = 0.5$ and we find $n_s = 1.04$ and $\alpha_s = -0.002$.

1) For n=4.

The spectral index and its running and the tensor-scalar ratio are

$$\begin{aligned}
n_s &= 1 + \left(\mu - \frac{3}{16}\right) r, \\
\alpha_s &= -\frac{3r^2}{256} (1 + 16\mu), \\
r &= \frac{16}{N}.
\end{aligned} \tag{26}$$

If we take $14 \leq N \leq 75$, $0.21 \leq r \leq 1.14$ and $-1.5 \times (1 + 16\mu) \times 10^{-2} \leq \alpha_s \leq -5.2 \times (1 + 16\mu) \times 10^{-4}$. In the commutative spacetime $\mu = 0$, the running of the spectral index satisfies $\alpha_s \in (-10^{-2}, -10^{-4})$. If we take $N = 14$ to fit $n_s = 1.21$, we find $r = 1.14$, $\mu = 0.372$ and $\alpha_s = -10^{-1}$. For $N = 50$ and $\mu = 0.5$, we find $n_s = 1.1$ and $\alpha_s = -0.01$.

The results are showed in fig 2. If we take the running spectral index into account, the chaotic inflations with potential $V \sim \phi^2$ and $V \sim \phi^4$ are on the boundary of the 2σ range if $\mu = 0.5$ and $N = 50$. If the number of the e-folds can be released to $N = 14$, these two chaotic inflation can not be ruled out at a level of about 2 standard deviations by WMAP three years data.

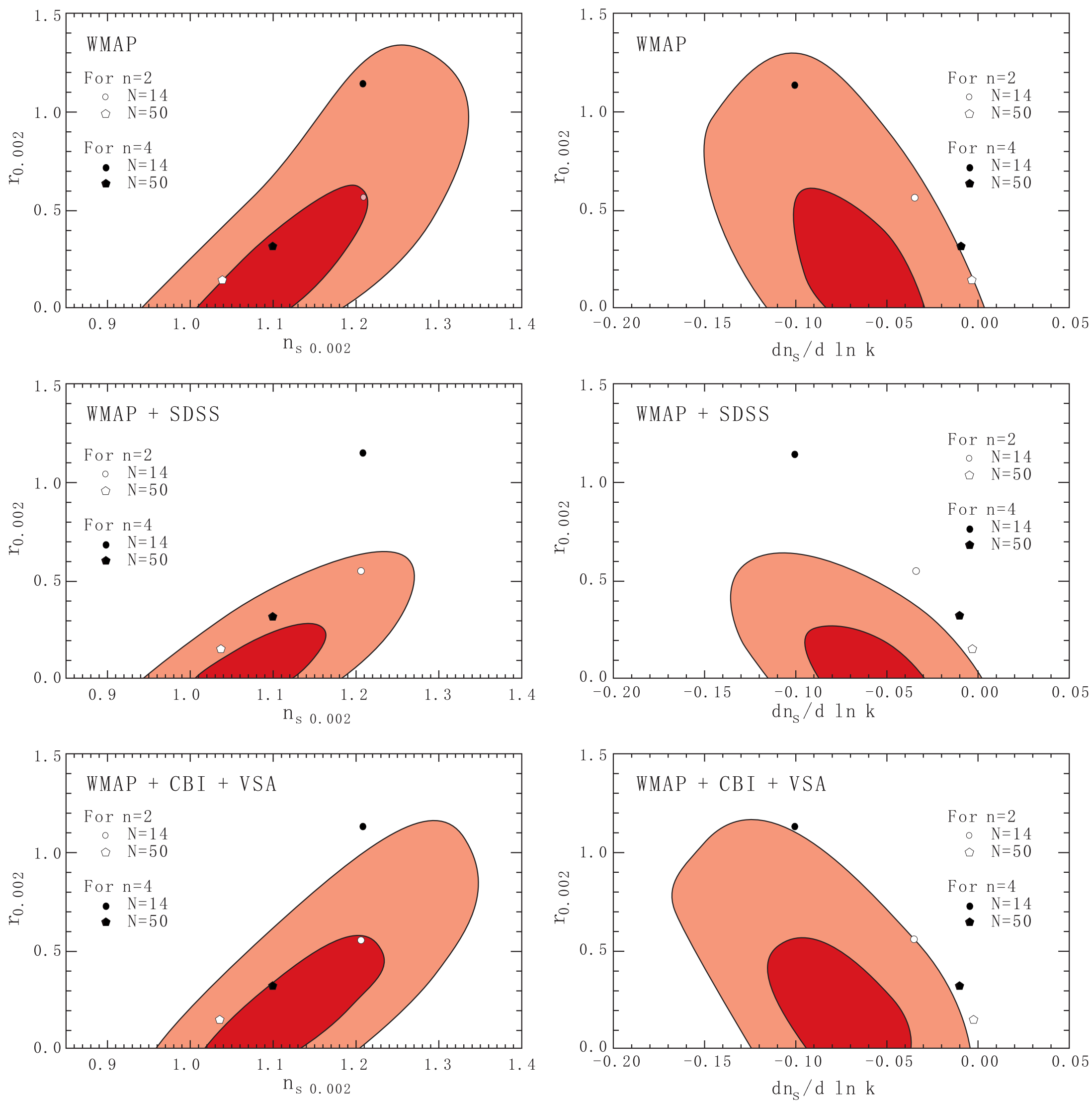

Figure 2. *With potential $V(\phi) \sim \phi^n$. N is the number of e-folds before the end of the inflation. Here we used the copy of Fig. 12 in [2].*

• *Power-law inflation*

Power-law inflation is drived by a scalar field with potential $V(\phi) = \lambda^4 \exp\left(-\sqrt{\frac{2}{n}}\frac{\phi}{M_p}\right)$. The scale factor can be exactly solved as

$$a(t) = \left(\frac{t}{(n+1)l}\right)^n, \tag{27}$$

here we re-parameterize λ to be l. The amplitude of the scalar power spectrum is given by (in [7,8])

$$\Delta_{\mathcal{R}}^2 = Bk^{-\frac{2}{n-1}}\left(1-\sigma\left(\frac{k_c}{k}\right)^{\frac{4}{n-1}}\right), \tag{28}$$

where

$$\begin{aligned} B &= \left(\frac{n(2n-1)}{(n+1)^2}\right)^{\frac{n}{n-1}} \frac{n}{8\pi^2}\left(\frac{l_p}{l}\right)^2 l^{-\frac{2}{n-1}}, \\ \sigma &= \frac{4n^2(n-2)(2n+1)}{(n+1)^2(n-1)(2n-1)}, \\ k_c &= \left(\frac{n(2n-1)}{(n+1)^2}\right)^{\frac{n+1}{4}} l_s^{-1}\left(\frac{l_s}{l}\right)^n, \end{aligned} \tag{29}$$

where $l_p = M_p^{-1} = 8.106 \times 10^{-33}$cm. The spectral index and its running are given by

$$n_s = 1 - \frac{2}{n-1} + \frac{4\sigma}{n-1}\left(\frac{k_c}{k}\right)^{\frac{4}{n-1}}, \tag{30}$$

$$\alpha_s = -\frac{16\sigma}{(n-1)^2}\left(\frac{k_c}{k}\right)^{\frac{4}{n-1}} \tag{31}$$

and the tensor-scalar ratio is

$$r = \frac{16}{n}. \tag{32}$$

Fitting the central value of the amplitude of power spectrum, spectral index and its running in (18), we find $n = 14.9$, $l = 3.05 \times 10^{-25}$ cm and $l_s = 3.85 \times 10^{-29}$ cm. Or equivalently, $M_s = l_s^{-1} = 2.1 \times 10^{-4} M_p$. We also find $k_c = 5.38 \times 10^{-5} Mpc^{-1}$ and the tensor-scalar ratio $r = 1.1$ which is within the allowed range of WMAP. Further we also calculate the spectral index and its running at $k = 0.05 Mpc^{-1}$ as $n_s = 0.996$ and $\alpha_s = -0.040$.

• *Small-field inflation*

This kind of model predicts a tiny tensor-scalar ratio. The spacetime noncommutative effects can not help us to improve the running spectral index significantly.

To conclude, the three year WMAP results reveal that the CMB power spectrum is not featureless, and a running spectral index is still alive and the model independent analysis shows that the noncommutative inflation model can explain the new results. However, in some typical inflation models, such as chaotic inflation, a rather low e-folding number is required and may appear unnatural. The power spectrum for the power-law inflation model becomes red at $k = 0.05Mpc^{-1}$ in our results which is consistent with WMAP. But the running seems a little too large than what we expect. Thus it is of great interest to have an investigation of a more general mechanism to generate a large running. We hope to return to this problem in the near future.

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